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A PROGRESS REPORT ON COMMUNICATIONS DIGITAL SIGNAL PROCESSING: THEORY AND PERFORMANCE OF FREQUENCY DOMAIN DIFFERENTIALLY ENCODED MULTI-FREQUENCY MODULATION

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ABSTRACT

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I INTRODUCTION

Multi-frequency Modulation

Multi-frequency modulation (MFM) is a digital modulation technique that is being researched at the Naval Postgraduate School for application in military digital communications systems. It is a high-data-rate, highly-bandwidth-efficient, modulation method with important potential for point-to-point, broadcast and packet switched network digital communications links. The method makes extensive use of digital signal processing techniques for its implementation.

MFM is also known as orthogonal frequency division multiplexing (OFDM), because of the orthogonality of the multiplicity of carrier frequencies that are used to transport the digital data. OFDM has been discussed in the literature in the context of data modems for the public switched telephone network (PSTN), as a groupband modem, as a mobile HF modem, and for satellite broadcast of digital audio programs(DAB)[1],[2],[3],[4]. One of the difficulties in the practical application of MFM (OFDM), is that of carrier recovery for demodulation. Unlike ordinary bandpass digital modulation techniques that employ a single carrier, MFM uses a great number, perhaps hundreds or thousands, of carriers simultaneously. If the channel contains any envelope delay, that is phase shift that is not linear with frequency, then the channel transfer function must be known, or estimated, accurately if the carrier phases are to be used to carry data.

During the past year we have concentrated on solving this problem. A solution that we have found works well in a practical channel [5], is to differentially encode information in the phase of adjacent tones in the same baud. The purpose of this report is to document the theoretical performance of this and of more general classes of MFM differential modulation.

The underlying principles of MFM generation and demodulation were reported last year [6]. The results of that report serve as a point of departure for this work. An MFM packet is shown in Figure

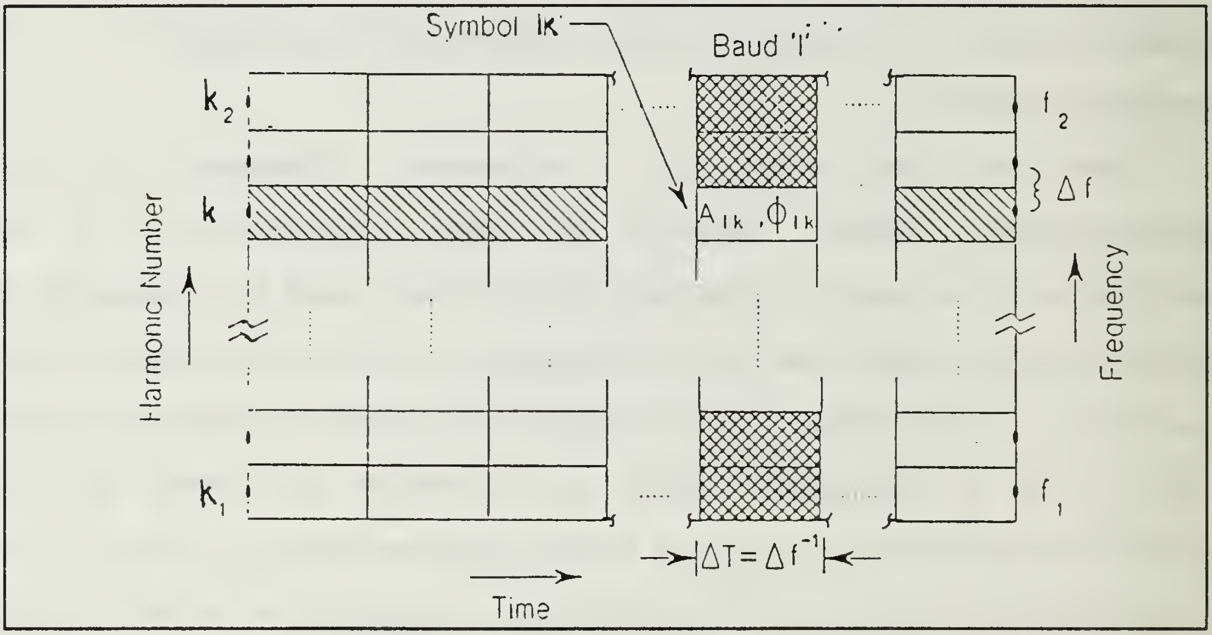

Figure 1 An MFM Packet

1. The terms used in this report are consistent with those of [6].

<u>Signal-to-Noise Ratio and E_b/N_o</u>

It was shown in our previous report [6, pg 24] that for any given baud corrupted by additive white gaussian noise(AWGN), the output of the DFT in the MFM receiver is the complex frequency domain sequence

$$Y(k) = S(k) + W(k) \qquad ; \quad k_1 \le k \le k_2 \tag{1.1}$$

where the received signal baud is given by

$$S(k) = A_k \cos\phi_k + jA_k \sin\phi_k , \tag{1.2}$$

and

$$A_k = k_x (2P_k)^{½}/2$$

with P_k the received tone power and k_x is the number of points in the DFT. W(k) is a white zero mean complex gaussian frequency domain sequence with uncorrelated real and imaginary parts that have equal variances $k_x^2 \Delta f N_o/4$. The frequency spacing between tones is Δf. By dividing (1.1) by the standard deviation of the noise components one obtains the unit variance complex sequence

$$F(k) = R(k) + jI(k) = S'(k) + W'(k) \tag{1.3}$$

with mean value

$$E[F(k)] = S'(k) = A'_k \cos\phi_k + jA'_k \sin\phi_k \tag{1.4}$$

where

$$A'_k = (2E_k/N_o)^{½} . \tag{1.5}$$

We have made use of the fact that $P_k/\Delta f = E_k$ is the received energy in the kth tone. This is because Δf is the reciprocal of the baud length ΔT.

In some applications, the channel signal-to-noise ratio is the parameter most pertinent to system performance. We shall refer to the signal-to-noise ratio in the signal band as the wideband signal-to-noise ratio, SNR_{WB}, and define it as the average signal power in the band to the average noise power in the same band. Thus

$$SNR_{WB} = KP/N_oW = E/N_o \tag{1.6}$$

where P and E are the average tone power and tone energy respectively, and K is the number of tones in the signal band of width $W = K\Delta f$. Furthermore, if there are Q bits transmitted per tone then $E_b = E/Q$ is the average energy per bit transmitted and

$$E_b/N_o = SNR_{WB}/Q = E/QN_o. \tag{1.7}$$

We shall use (1.5) through (1.7) freely when making performance comparisons of various types of MFM modulation.

MFM Constellations

Three classes of modulation will be considered in this report; phase modulation (PSK), amplitude modulation (ASK) and combined amplitude and phase modulation. We shall refer to the latter, hopefully unambiguously, as QAM. (Ambiguity could arise from the fact the the PSK is coded and decoded using its quadrature components, but the context of our discussions should make the distinction between QAM and straight PSK clear.) In each case we shall present results for both differential and non-differential encoding. A particuarily useful digital modulation format is a hybrid: differential phase coding and non-differential amplitude coding. The hybrid constellation for MFM has been developed by Mercury Digital Communications, Inc.[7].

Differential modulation is obtained by encoding the input data symbols as changes in signal state occupancy of adjacent tone constellations in such a way that the changes are invariant under a rotation of the constellations. The principal purpose for differential encoding is to render the transmission insensitive to uncompensated channel phase shift. In MFM the tones can be placed very close together. If adjacent tones are close enough together then their uncompensated phase, that is the rotation of their constellations, will be nearly identical, and it cancels out in the decoding.

II ERROR PATTERNS AND PERFORMANCE MEASURES

Symbols and Bits

In MFM, Q bit symbols are coded into the phase and amplitude of each of the multiplicity of carrier frequency tones during each baud. The symbols are partitioned into M bits that are coded into tone phase and N bits that are coded into tone amplitude (Q=M+N). We adopt the notation

$$\underline{s} = \underline{p},\underline{a} \tag{2.1}$$

to designate the symbol and its partition into phase and amplitude bits. For example, with 4 bits per tone, one may have

$$1010 = 101,0 \tag{2.2}$$

and three bits of phase, in this example 101, are coded as one of eight possible tone phases and one bit of amplitude, in this example 0, is coded as one of two possible tone amplitudes.

Errors may occur in the receiver in any of the Q bit positions. An error pattern is defined as a Q bit symbol

$$\underline{e} = \underline{e}_p,\underline{e}_a \tag{2.3}$$

which we also partition into phase bit errors and amplitude bit errors. There are 2^Q total error patterns made up from 2^M phase error patterns for each of the 2^N amplitude error patterns.

The output of the receiver is the decoded symbol

$$\underline{d} = \underline{s} + \underline{e} \tag{2.4}$$

where the plus sign designates bit by bit binary addition (logical "or") of the transmit symbol with the error pattern. The decoded symbol is also partitioned into decoded phase and amplitude bits,

$$\underline{d} = \underline{d}_p, \underline{d}_a \tag{2.5}$$

A symbol error occurs anytime that $\underline{d} \neq \underline{s}$, that is for any of the 2^Q-1 non-zero error patterns, $\underline{e} \neq \underline{0}$.

<u>Symbol Error Probabilities and Bit Error Rate (BER)</u>

The probability of symbol error is

$$P_s = \sum_i \Pr[\underline{e} \neq \underline{0} \mid \underline{s} = \underline{s}_i] \Pr[\underline{s} = \underline{s}_i] \quad . \tag{2.6}$$

If we assume proper source coding then each of the 2^Q data symbols will be equally probable and

$$P_s = 2^{-Q} \sum_i \Pr[\underline{e} \neq \underline{0} \mid \underline{s} = \underline{s}_i] . \tag{2.7}$$

Performance of digital communications systems is customarily specified is terms of bit error rate (BER). BER is defined as the expected number of bit errors in a very long sequence of symbols divided by the number of bits in the sequence. For example, if the expected number of bit errors in a sequence of one million bits is one, then the BER is 10^{-6}. As long as symbols that make up a sequence of bits are statistically independent and the error probabilities are stationary, then the BER is also the expected number of bit errors in any given received symbol divided by Q, the number of bits in the symbol. We shall adopt this definition of BER in this report.

Let B be the number of symbol bits in error, that is the number of ones in $\underline{e}$. Then

$$E[B] = \Sigma\ E[B|\underline{s}=\underline{s}_i]\Pr[\underline{s}=\underline{s}_i] \qquad (2.8)$$

and again assuming equally probable symbols,

$$E[B] = 2^{-Q}\ \Sigma\ E[B|\underline{s}=\underline{s}_i] \qquad (2.9)$$

with

$$BER = E[B]/Q \qquad (2.10)$$

in either case.

Single and Multiple Bit Errors

BER is a normalized expected value. Sometimes it is most convienent to compute it as a weighted superposition of terms corresponding to single bit errors, double bit errors, etc, up to Q bit errors. That is,

$$E[B] = \sum_{1}^{Q} b \Pr[B=b] = 2^{-Q} \sum_{i} \{\sum_{1}^{Q} b \Pr[B=b|\underline{s}=\underline{s}_i]\} \qquad (2.11)$$

where, $\Pr[B=b|\underline{s}=\underline{s}_i]$ are the conditional bit error probabilities. These can be computed from the conditional error pattern probabilities as follows;

$$\Pr[B=b|\underline{s}=\underline{s}_i] = \sum \Pr[\underline{e}|\underline{e} \text{ contains b ones}, \underline{s}=\underline{s}_i] \qquad (2.12)$$

where the sum is over the $N_b = (Q,b)$ error patterns containing b ones out of Q bits. For example, there are $N_1 = (Q,1) = Q$ single bit error patterns, $N_2 = (Q,2) = Q(Q-1)/2$ double bit error patterns etc.

In summary, the symbol error probability is computed from (2.7) and the BER from (2.10),(2.11) and (2.12). Both computations require the conditional error pattern probabilities $\Pr[\underline{e}=\underline{e}_j|\underline{s}=\underline{s}_i]$. The objective of the next three Chapters is to obtain theoretical expressions for these for various MFM and MFDM constellations.

III PHASE MODULATION

In this Chapter MFM and MFDM systems that use constant amplitudes and employ only phase modulation to encode the Q symbol bits (M=Q, N=0) will be considered. Two types of modulation will be analyzed; coherent modulation/demodulation and differential modulation/demodulation. In coherent modulation, the absolute phase of each of the carrier tones must be known at the receiver. This form of modulation should only be used in a linear phase channel, or in a channel that has been equalized to linear phase. Any residual phase jitter will affect performance as noise, and any significant unequalized non-linear phase (group delay) will most probably render coherent phase modualtion unworkable at any SNR. However, its analysis, assuming perfect phase equalization, is included for comparison with differential modulation.

Differential phase modultion of MFM, MFDM, encodes phase bits as the phase difference between adjacent carrier frequency tones in the same baud. Although it is not necessary that adjacent tones be used, or that the same baud be used (we could encode using the phase difference of each of the same tones on successive bauds as is done in conventional single carrier differential phase modulation (DPSK) transmission), the theory behind differential encoding in the frequency domain using adjacent tones is that the differential phase jitter between adjacent tones introduced by the channel can be reduced to any desired level by putting the tones sufficiently close together in frequency. Recall that the tone spacing in MFM is $\Delta f = 1/\Delta T$. Therefore differential phase shift through an arbitrary unequalized channel is inversely proportional to baud length.

Coherent Phase Modulation/Demodulation

A phase constellation for 2^M-ary PSK is generated by utilizing a constant tone amplitude and 2^M phases. The phases are equally spaced at intervals $\Delta\phi = 2\pi/2^M$, then rotated by adding $\Delta\phi/2$. Thus the modulation phase bits are encoded according to the transformation,

$$\underline{p(k)}_i \rightarrow d\phi(k) = i(k)\Delta\phi \quad ; \quad i = 0,1,\ldots.2^M-1$$
$$\phi(k) = d\phi(k) + \Delta\phi/2 \tag{3.1}$$

Such a constellation for 8 phases is shown in Figure 2.

The maximum likelihood demodulation rule (MLD) is to decode the complex point Y(k) generated by the receiver DFT(the matched filter receiver for MFM, see [6]) to the nearest point of the constellation. The decoding boundaries for MLD are sectors of anglular width $\Delta\phi$ as illustrated for an 2^M of eight in Figure 2. We shall refer to this type of decoding as "sector decoding" (SD). Notice that SD is independent of tone amplitude A, or equivalently of E/N_0. This makes the implementation of SD in the receiver independent of system gain, a highly desireable feature. Of course system performance is not independent of SNR.

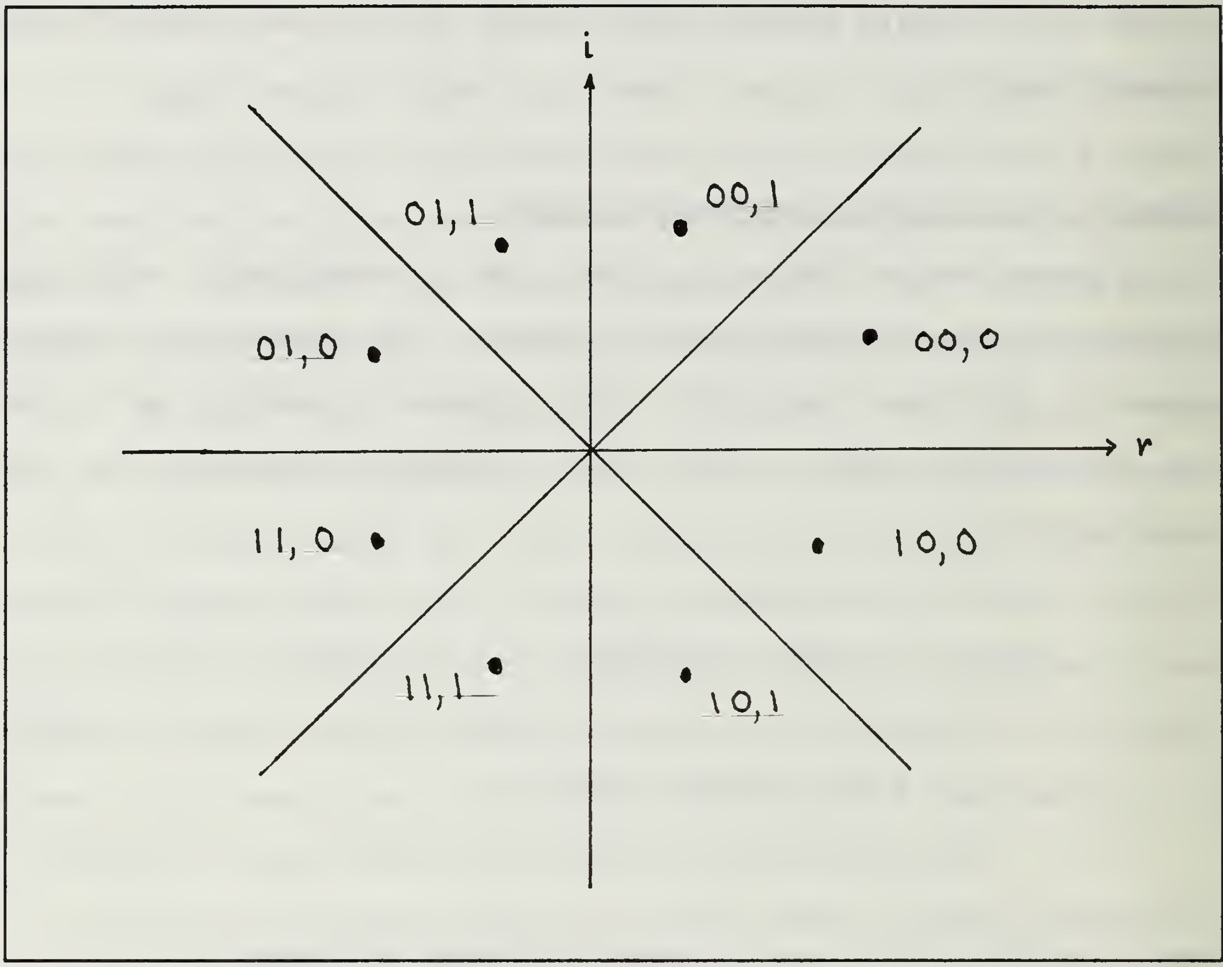

Figure 2 An Eight PSK Constellation with Sector Decoding Boundaries

a. Symbol Error Probability

Phase symbol error decoding probabilities are the same for each of the 2^M symbols. This is so because the statistical properties of the complex noise W(k) are invariant to phase rotations. That is, the real and imaginary parts of W(k)exp(jϕ) are, like those of W(k), uncorrelated, equal variance gaussian random variables. Consequently, the probability of symbol decoding error given by (2.7) is the same as the conditional probability of decoding error

$$P_s = \Pr[\underline{e} \neq \underline{0} \,|\, \underline{s} = \underline{s}_i], \text{ for any } i. \tag{3.2}$$

The probability that $\underline{e} \neq \underline{0}$ when $\underline{s} = \underline{s}_i$ is the probability that Y(k), or equivalently F(k), is not in the sector of angular width $\Delta\phi$ defined by the angles $i\Delta\phi$ and $(i+1)\Delta\phi$.

i.) QPSK (M = 2)

We include this case for completeness as the results are well known and were presented in the previous report[6]. Also, for more phase states and narrower phase sectors, the results that will be presented in the next section are only approximate(the approximation becomes more accurate as the sectors become more narrow). Exact results are easily found for QPSK as follows:

The probability that F(k) is not in the first quadrant, given $\underline{s}_0$ is simply one minus the probability that F(k) is in the first quadrant. The probability that F(k) is in the first quadrant is the probability that $R(k) \geq 0$ and that $I(k) \geq 0$. Since the real and imaginary parts of F(k) are statistically independent;

$$P_s = 1 - \Pr\{R(k) \geq 0\}\Pr\{I(k) \geq 0\}. \tag{3.3}$$

These probabilities are easily found using the fact that R(k) and I(k) are unit variance gaussian random variables with mean values determined from (1.4)-(1.6) to be

$$E\{R(k)\} = E\{I(k)\} = A'_k/(2)^{1/2} = \rho \qquad (3.4)$$

with

$$\rho = \{SNR_{WB}\}^{1/2} = \{E/N_o\}^{1/2} ,$$

where we have used the fact that $\phi_k = \pi/4$ for $\underline{s} = \underline{s}_0$, the symbol assumed as transmitted on tone k. Consequently,

$$P_s = 1 - [1 - Q(\rho)]^2 \approx 2Q(\rho) \qquad (3.5)$$

is the symbol error probability. The approximation is very good for SNR_{WB} greater than about 10 db. Note from (1.7) that for QPSK

$$\rho^2 = E/N_o = SNR_{WB} = 2E_b/N_o. \qquad (3.6)$$

ii.) 2^M-ary PSK

Let $p(\Delta\theta)$ be the probability that F(k) falls outside an angular sector of width $\pm\Delta\theta/2$ with respect to ϕ_i, the angle of F(k) given the transmission of $\underline{s}_i$ in the absence of noise. That is,

$$p(\Delta\theta) = \Pr[|\alpha - \phi_i| \geq \Delta\theta/2] \qquad (3.7)$$

where α is the angle of Y(k) (or equivalently of F(k)). It is known that for $\Delta\theta \leq \pi/2$, this probability is well approximated by ([8]; Hakin pg 317).

$$p(\Delta\theta) \approx 2Q\{A'_k \sin(\Delta\theta/2)\} = 2Q\{(2)^{1/2}\rho\sin(\Delta\theta/2)\} \quad (3.8)$$

and that the approximation becomes increasingly more accurate with increasing ρ and decreasing $\Delta\theta$. A PSK symbol error occurs if $|\alpha - \phi_i| \geq \Delta\phi/2$. Therefore, the PSK symbol error probability is

$$P_s = p(\Delta\phi) \approx 2Q\{(2)^{1/2}\rho\sin(\Delta\phi/2)\}. \quad (3.9)$$

We note that (3.9) with $\Delta\phi = \pi/2$, which is QPSK, agrees with the approximation generated by binomial expansion of the exact equation derived for QPSK and presented in (3.5).

Differential Phase Modulation/Demodulation

In MFDPSK, the phase bits for tone k are encoded into equally spaced phases

$$\underline{p(k)}_i \rightarrow d\phi(k) = i(k)\Delta\phi \quad ; \; i=1,2,\ldots.2^M-1, \quad (3.10)$$

and this differential phase shift is added to the phase of the previous tone to obtain the tone phase of tone k as

$$\phi(k) = \phi(k-1) + d\phi(k). \tag{3.11}$$

An initial condition must be chosen. By choosing the phase of the first carrier frequency tone as

$$\phi(k_1) = \Delta\phi/2 \tag{3.12}$$

the constellations of all tones are rotated by $\Delta\phi/2$ thereby yeilding transmit tone phases identical to those used in coherent MFPSK discussed in the previous section.

Differential decoding the phase bits is accomplished by computing the complex rotated product

$$D(k) = F(k)F^*(k-1)\exp[j\Delta\phi/2] \; ; \qquad k_1+1 \le k \le k_2 \tag{3.13}$$

for adjacent digital frequencies in each MFDM baud. In the absence of noise, the phase of D(k) is the phase difference between tone k and tone k-1 plus $\Delta\phi/2$, that is $i(k)\Delta\phi + \Delta\phi/2$. Comparison with (3.1) reveals that the complex rotated products have exactly the same constellations as described for coherent PSK. The differential receiver optimally decodes the phase bits at this stage by using the same sector decoding rules described above.

a. Statistical properties of the Complex Rotated Product.

Define the complex random product

$$F_p(k) = F(k)F^*(k-1) = S'(k)S'^*(k-1) + W_p(k) \tag{3.14}$$

with

$$W_p(k) = W'(k)W'^*(k-1) + S'(k)W'^*(k-1) + W'(k)S'^*(k-1). \quad (3.15)$$

Since $W'(k)$ is zero mean, it follows directly that $W_p(k)$ has zero mean and therefore that the mean of $F_p(k)$ is

$$E[F_p(k)] = S'(k)S'^*(k-1) = \{2(E_k E_{k-1})^{1/2}/N_o\}\exp[jd\phi(k)]. \quad (3.16)$$

The real and imaginary parts of the complex product are

$$\begin{aligned} R_p(k) &= \mathrm{Re}[F(k)F^*(k-1)] = R(k)R(k-1) + I(k)I(k-1) \\ I_p(k) &= \mathrm{Im}[F(k)F^*(k-1)] = I(k)R(k-1) - R(k)I(k-1). \end{aligned} \quad (3.17)$$

Because of the statistical independence between adjacent tones and between the real and imaginary parts of the same tone, it follows directly that

$$\begin{aligned} \mathrm{Var}\{\mathrm{Re}[W_p(k)]\} &= \mathrm{Var}\{R_p(k)\} = \mathrm{Var}[R(k)R(k-1)] + \mathrm{Var}[I(k)I(k-1)] \\ \mathrm{Var}\{\mathrm{Im}[W_p(k)]\} &= \mathrm{Var}\{I_p(k)\} = \mathrm{Var}[I(k)R(k-1)] + \mathrm{Var}[R(k)I(k-1)]. \end{aligned} \quad (3.18)$$

Let X and Y be statistically independent, non-zero mean random variables. Then

$$Var[XY] = E[X^2Y^2] - \{E[XY]\}^2 =$$

$$E[X^2]E[Y^2] - \{E[X]\}^2\{E[Y]\}^2 =$$

$$\sigma_x^2\sigma_y^2 + \{E[X]\}^2\sigma_y^2 + \{E[Y]\}^2\sigma_x^2. \qquad (3.19)$$

It can be seen that the variance of a product contains a noise times noise term and two signal times noise terms; the later two are due to the mean values of X and Y. Recalling now that the real and imaginary values R(k) and I(k) have been normalized to unit variance for all k, then it follows that

$$\sigma_p^2 = Var\{R_p(k)\} = 2(1 + E_k/N_o + E_{k-1}/N_o)$$

$$(3.20)$$

$$\sigma_p^2 = Var\{I_p(k)\} = 2(1 + E_k/N_o + E_{k-1}/N_o).$$

Here, the first term is due to noise times noise, the second two are the signal times noise terms. This is important because at high SNR, the signal times noise terms dominate and R_p amd I_p are approximately gaussian with mean values given by (3.16) and identical variances given by (3.20).

It is important to determine if the real and imaginary parts of the complex product noise terms, $W_p(k)$ are correlated. We note that, because $W_p(k)$ is zero mean,

$$Covar\{R_p(k)I_p(k)\} = E\{Re[W_p(k)]Im[W_p(k)]\}. \qquad (3.21)$$

The covariance is defined as

$$\mathrm{Covar}\{R_p(k)I_p(k)\} = E\{R_p(k)I_p(k)\} - E\{R_p(k)\}E\{I_p(k)\}. \quad (3.22)$$

which we find by using (3.17) and the previously cited properties of the noise to be identically zero(this is so at all SNRs and is not dependent on the gaussian approximation). Zero covariance insures that D(k), the rotated complex product of (3.13) also has uncorrelated real and imaginary parts with equal variances as given by (3.20). At high SNRs it tends to the gaussian.

Dividing (3.13) by σ_p produces a normalized differentially demodulated complex random variable D'(k) with unit variance real and imaginary parts and mean value given by,

$$E[D'(k)] = \{S'(k)S'^{*}(k-1)/\sigma_p\}\exp\{j\Delta\phi/2\} =$$

$$M'_k\cos(d\phi(k)+\Delta\phi/2) + jM'_k\sin(d\phi(k)+\Delta\phi/2) \quad (3.23)$$

where

$$M'_k = [(2E_kE_{k-1})^{1/2}/N_o]/[1+E_k/N_o+E_{k-1}/N_o]^{1/2}. \quad (3.24)$$

Comparing (3.23) and (3.24), with (1.4) and (1.5) it can be seen that differential modulation causes a reduction in SNR for decoding. If, for example, adjacent tones have equal energy and if the $SNR_{WB} = E_k/N_o$ is much larger than one, then

$$M'_k \approx \{E_k/N_o\}^{1/2} = \rho_k \qquad (3.25)$$

which is 3 db less than (1.5) predicts for the coherent PSK.

The symbol error probabilites are determined as before. So long as the SNR is sufficiently high to use the gaussian approximation, then the probability of a symbol error for MFDQPSK is

$$P_s = 1 - [1 - Q\{M'_k/(2)^{1/2}\}]^2 \approx 2Q\{M'_k/(2)^{1/2}\}. \qquad (3.26)$$

The probability that D(k) falls outside a sector of width $\Delta\theta$ is

$$p(\Delta\theta) \approx 2Q\{M'_k \sin(\Delta\theta/2)\} \qquad (3.27)$$

so the probability of a differential symbol decoding error in 2^M-ary MFPSK is

$$P_s = p(\Delta\phi) \qquad (3.28)$$

with M'_k given by (3.24) and $\Delta\phi = 2\pi/2^M$.

Bit Error Rates for MFPSK and MFDPSK

In order to compute BERs, the expected number of bit errors per symbol must be computed and divided by the number of bits per symbol (see (2.10)). The probability of single, double, triple bit errors etc. depends on the transformation that is used , that is it depends on (3.1),(3.10),

$$\underline{p(k)}_i \rightarrow i(k)\Delta\phi. \qquad (3.29)$$

Because the most probable errors are in the sectors that are adjacent to the sector containing the transmitted symbol, and the next most probable are in the next set of adjacent sectors, etc., we have elected to use a reflex or Grey code to code the phase bits. In this way, adjacent sectors only differ by one bit, so that the most probable symbol decoding errors are single bit errors, next most adjacent sectors differ by two bits, so they yield double bit errors. In this way, we hope to minimize BER when the errors are due to AWGN.

The Grey reflex code for 2, 3, and 4 phase bits is given for reference in Table 3,1

$\underline{p(k)}_i$			$i(k)$		
M=2	M=3	M=4	M=2	M=3	M=4
00	00,0	00,00	0	0	0
		00,01			1
	00,1	00,11		1	2
		00,10			3
01	01,1	01,10	1	2	4
		01,11			5

	01,0	01,01		3	6
		01,00			7
11	11,0	11,00	2	4	8
		11,01			9
	11,1	11,11		5	10
		11,10			11
10	10,1	10,10	3	6	12
		10,11			13
	10,0	10,01		7	14
		10,00			15

TABLE 3.1 Reflex Coded Phase Bits

Note from Figure 2 that the two most significant bits are determined soley from the quadrant in which the received complex point occurs. The remaining bits depend only on the sector occupied by the absolute values of the real and imaginary parts of the complex values. These observations are useful for designing encoding and decoding algorithims for variable numbers of bits of phase modulation in a systematic fashion (see, Mercury Digital Communications patent application [7]).

a. MFQPSK and MFDQPSK Bit Error Rates.

It is easy to see that the two quad bits are decoded as follows;

Most Significant Bit $\rightarrow$ 0 if $I(k) \geq 0$;

Most Significant Bit $\rightarrow$ 1 if $I(k) < 0$.

Next Most Significant Bit $\rightarrow$ 0 if $R(k) \geq 0$;

Next Most Significant Bit $\rightarrow$ 1 if $R(k) < 0$.

Here R(k) and I(k) are the real and imaginary parts of Y(k) or of D(k) for coherent MFQPSK or MFDQPSK respectively. Because, in both cases they are statistically independent random variables, then QPSK is equivalent to sending two independent one bit symbols and consequently the BER is just the symbol(bit) error probability of either one. Thus, for QPSK

$$BER = Q(x) \tag{3.30}$$

where

for Coherent MFQPSK;

$$x = A'_k/(2)^{1/2} = \rho \tag{3.31}$$

and for MFDQPSK;

$$x = M'_k/(2)^{1/2} = \rho/2^{1/2} \tag{3.32}$$

b. 2^M-ary Coherent MFPSK and MFDPSK.

Due to the symmetry of the phase sectors and the Grey coding algorithim employed in the PSK and DPSK, it is clear that the conditional probabilities of B bit errors in a symbol of M phase bits are all equal. That is, $Pr[B=b|\underline{s} = \underline{s}_i]$ is the same for all $\underline{s}_i$. and therefore so is $E[B|\underline{s} = \underline{s}_i]$. Thus, for PSK systems

$$BER = E[B|\underline{s}]/M \tag{3.33}$$

and

$$E[B|\underline{s}] = 1\{Pr[B=1|\underline{s}]\} + 2\{Pr[B=2|\underline{s}]\} + \ldots + M\{Pr[B=M|\underline{s}]\}. \tag{3.34}$$

Figure 3 illustrates B , the number of bit errors for M =4 in each of the 16 sectors with respect to the transmitted symbol $\underline{s}$ = 00,00. Note that of the 4 single bit errors, 2 are in adjacent sectors, and of the 6 double bit errors, 2 are in sectors adjacent to the single bit sectors. We shall assume for the remainder of this report that the probability a received complex value is more than two sectors away from the transmit sector is negligible compared with the probability of being within the two closest sectors. Thus,

$$Pr[B=1|\underline{s}] \approx p(\Delta\phi) - p(3\Delta\phi) \tag{3.35}$$

$$\Pr[B=2|\underline{s}] \approx p(3\Delta\phi)$$

and consequently

$$\text{BER} \approx \{p(\Delta\phi) + p(3\Delta\phi)\}/M. \qquad (3.36)$$

with $p(\Delta\theta)$ determined by (3.8) for coherent MFPSK and by (3.27) for MFDQPSK.

Equations (3.30) and (3.36) represent the principal results of Chapter III. They are used to compute BER for coherent phase coded and for differentially phase coded MFM. Grey coding of the phase bits into the phase sectors has been assumed.

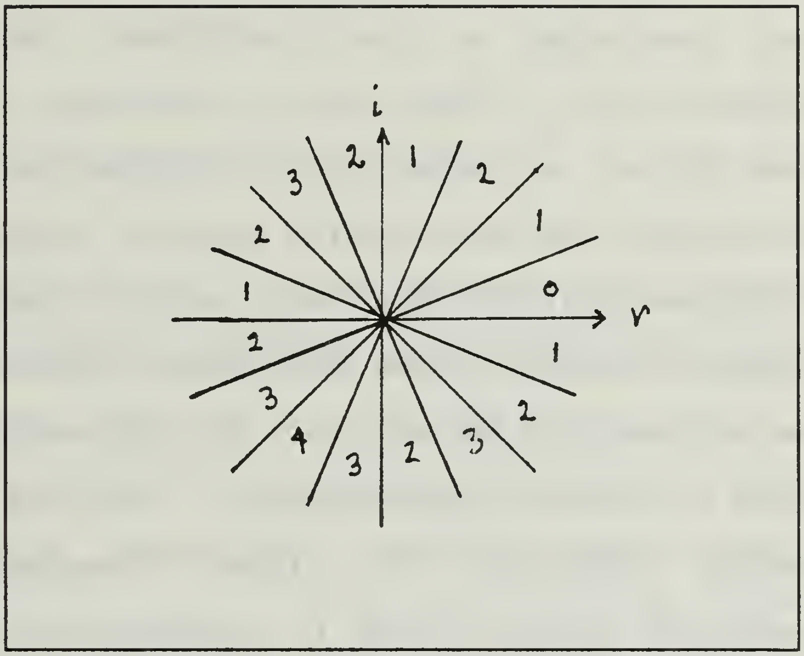

Figure 3 Distribution of Bit Errors B for $\underline{s}$ =00,00.

IV AMPLITUDE MODULATION

In this Chapter, results are presented for digitally coding information into the amplitude of MFM carrier tones. For the non-differential case, these are but a recitation of well known results for incoherent amplitude shift keying since all the tones are orthogonal and are in effect processed in parallel. The advantage of amplitude modulation is that amplitude coding does not require any knowledge of the individual carrier tone phases for demodulation. It does require knowledge or control of the gain of the channel at each of the carrier tones in order to set the thresholds for the decision regions. Consequently, a technique for differentially encoding data as the change in amplitude between adjacent carrier tones will also be described and analyzed. This method and an apparatus for its implementation have been described in a US Patent Application[7]. So long as the channel gain is nearly identical at adjacent frequencies, the differential amplitude coding system is independent of both the phase and the amplitude response of the channel.

Multilevel Amplitude Modulation

Multilevel, multi-frequency amplitude modulation encodes N amplitude bits into one of 2^N amplitude levels in each of the carrier tones. The coding is prescribed by a transformation;

$$\underline{a(k)}_i \rightarrow A_k = i(k)\Delta A + A_{min} \; ; \; i = 0,1...2^N-1, \quad (4.1)$$

and as with the MFPSK coding, a Grey code is used so that adjacent amplitude values correspond to only one bit difference in codewords.

Given the transmission of codeword $\underline{a(k)}$, the normalized decoded complex DFT coefficient F(k) is a unit variance gaussian complex random variable with uncorrelated real and imaginary parts and average values given by (1.5). The magnitude of F(k), is a Rice-Nakagami random variable, R, with probability density function

$$f_R(r|\underline{a(k)}) = r\exp\{-(r^2+A'_{k2})/2\}I_o(rA'_k) \tag{4.2}$$

Now, for $A'_k >>1$, integrals of $f_R(r)$ in the vicinity of A'_k can be approximated by integrals under the gaussian distribution with mean value A'_k and unit variance. (see e.g. [9], pg 504 and pg 625.)

Amplitude bits will be decoded in accordance with the MLD rule as illustrated in Figure 4. Conditional amplitude symbol error probabilities are;

$$P_s[A'_k = A'_{min}] \approx Q(\Delta A'/2)$$

$$P_s[A'_{min} < A'_k < A'_{max}] \approx 2Q(\Delta A'/2) \tag{4.3}$$

$$P_s[A'_k = A'_{max}] \approx Q(\Delta A'/2)$$

Assuming equally likely symbols, the average symbol errror probability is

$$P_s = 2^{-N} \Sigma P_s[A'_k] = 2(1- 2^{-N})Q(\Delta A'/2) \quad (4.4)$$

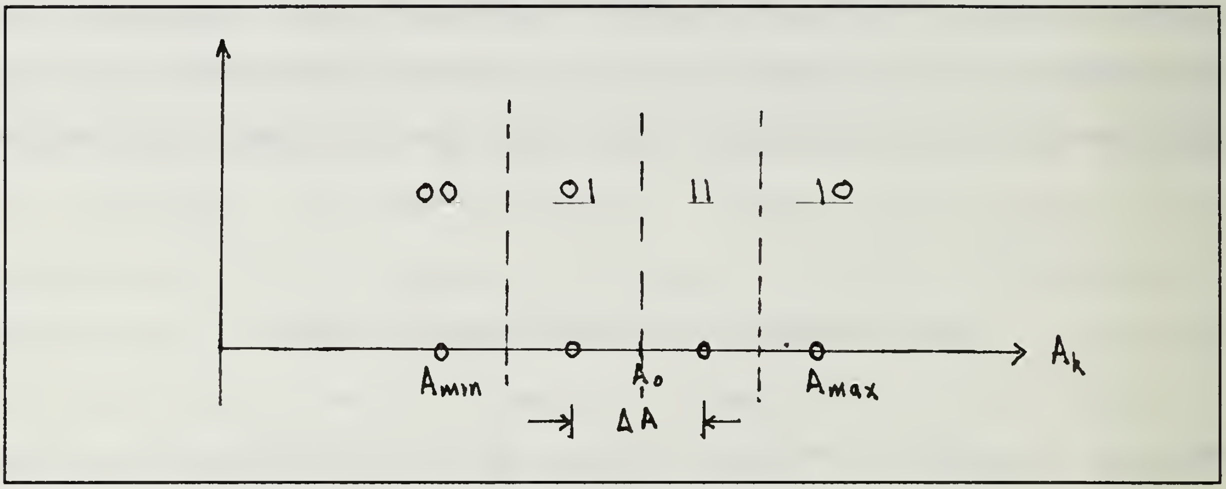

Figure 4 Multi-Level Amplitude Decoding

Average SNR_{WB}, E/N_o and E_b/N_o

When amplitude modulation is used, the carrier tones have different power. However, we would like to evaluate the error probabilites in terms of an average SNR_{WB} and average E_b/N_o. Let $A'_o = E[A_k']$ be the average signal amplitude of tone k and let the increment between amplitudes be proportional to the average amplitude, that is let $\Delta A' = \alpha A'_o$. Then it is easily shown that the average tone energy E is given by

$$E = E[E_k] = E_o\{1 + (\alpha/2)^2(2^N-1)[1+(2^N-2)/3]\} \quad (4.5)$$

where

$$E_o = (A'_o)^2 N_o/2 \qquad (4.6)$$

is the energy associated with the average amplitude of tone k. We shall be most interested in zero, one, two and three bits of amplitude coding for which N is zero, one, two and three respectively and (4.5) shows that the average energy is related to the energy of the average amplitude tone by

$\underline{N=0}$: $E = E_o$

$\underline{N=1}$: $E = E_o\{1 + (\alpha/2)^2\}$

(4.7)

$\underline{N=2}$: $E = E_o\{1 + 5(\alpha/2)^2\}$

$\underline{N=3}$: $E = E_o\{1 + 21(\alpha/2)^2\}$.

The average SNR_{WB} and average E_b/N_o are related to the average tone energy E as in (1.6) and (1.7), viz;

$$SNR_{WB} = E/N_o = \rho \qquad (4.8)$$

and

$$E_b/N_o = E/QN_o. \qquad (4.9)$$

In order to evaluate symbol error probabilites, we need to evaluate $Q(\Delta A'/2)$. Now recalling that $\Delta A'$ has been defined as a fraction of the average tone amplitude, then it follows that

$$Q(\Delta A'/2)=Q(\alpha A'_o/2) = Q(\alpha\{2E_o/N_o\}^{1/2}/2)$$

$$= Q(\{2E/N_o\}^{1/2}f(\alpha,N)) \qquad (4.10)$$

with

$$f(\alpha,N) = \alpha/[2\{1+(\alpha/2)^2(2^N-1)[1+(2^N-2)/3]\}^{1/2}] \qquad (4.11)$$

It is clear that the function $f(\alpha,N)$ is of central importance for controlling error probabilites. Notice that there is a maximum value for α because the smallest amplitude level must be non-negative and that the maximum value decreases with increasing N. That is,

$$0 < \alpha < 2/(2^N-1). \qquad (4.12)$$

We note that $f(\alpha,N)$ can be reasonably approximated by $\alpha/2$ for values of α less than one. It is also clear that if one is just using amplitude modulation, α should be made as large as possible, which is the same as spreading the tone amplitude levels as much as possible, in order to minimize error probabilities. Consequently, with amplitude modulation alone, performance is limited by

$$f(\alpha_{max},N) = \{1/(2^N-1)\}\{1/[1+(1/(2^N-1))(1+(2^N-2)/3)]^{1/2}\} \qquad (4.13)$$

For one, two, three and four bits of amplitude modulation, (4.13) is given in Table 4.2 below.

$\underline{N}$	$\underline{f(\alpha_{max},N)}$
1	$1/2^{1/2}$
2	$1/14^{1/2}$
3	$1/70^{1/2}$
4	$1/310^{1/2}$

TABLE 4.2

The Maximum of the Spread Function

Finally, combining (4.4) and (4.10), the symbol error probability for amplitude modualtion alone of the tones is given by

$$P_s = 2(1-2^{-N})Q\{(2)^{1/2}\rho f(\alpha,N)\}. \tag{4.14}$$

Amplitude and phase modulation symbol error probabilites for different numbers of bits may be compared by comparing (4.14) with (3.9).(It should be noted that for the situation of $\alpha = \alpha_{max}$, that the minimum amplitude $A'_{min}=0$ and therefore the probability distribution is Rayliegh, not Gaussian as assumed in (4.3). Since we are primarily interested in combined amplitude and phase modulation constellations for MFM for which 0 amplitude tones are not permissible, we shall not be concerned with presenting the necessary correction here.)

BER's for Amplitude Modulation

Bit error rates for the amplitude bits are determined as presented in Chapter II. That is,

$$BER = E[B]/N \qquad (4.15)$$

with

$$E[B] = \Sigma\, b\, \{\Sigma\, Pr[B=b|\underline{a}]Pr[\underline{a}]\} \qquad (4.16)$$

We shall approximate (4.16) using single and double bit errors. This is a good approximation by virtue of the Grey coding of the amplitudes bits. Thus, assuming equally likely amplitude codewords,

$$E[B] = 2^{-N}\, \{\Sigma\, Pr[B=1|\underline{a}] + 2\, \Sigma\, Pr[B=2|\underline{a}]\} \qquad (4.16)$$

These conditional error probabilities depend on the whether, the codeword $\underline{a}$ corresponds to A'_{min} or A'_{max} or to an in-between value of A'. We are principally concerned with one, two and three bits of amplitude modulation. The evaluation of (4.16) for these cases is the following:

N=1:

$$E[B] = Ps = Q(\Delta A'/2) = BER \qquad (4.17)$$

N=2:

$$E[B] = 1.5Q(\Delta A'/2) + Q(3\Delta A'/2) = 2BER \qquad (4.18)$$

N=3:

$$E[B] = (7/4)Q(\Delta A'/2) + (3/2)Q(3\Delta A'/2) = 3BER \quad (4.19)$$

with

$$\Delta A'/2 = \{2\}^{1/2}\rho f(\alpha,N) \quad (4.20)$$

Differential Amplitude Modulation

In order to decode the amplitude bits of MFM carrier tones, it is necessary to establish accurate values for the channel gain at each carrier tone frequency. Although this is less difficult to do than to determine phase shift, it may still be of advantage to have a modulation scheme which does not require either phase or amplitude information in order to decode the symbols. In mobile systems with bandpass signals subject to fading this will almost certainly be the case. Also, in switched systems where a premium is on connect time, allocation of valuable connection time to measurement of channel gain may represent a limitation. Consequently, we have investigated the performance of MFM modulation with differentially encoded amplitude information. Although encoding more than one bit differentially is conceptually possible, to date we have only analyzed and tested modulation schemes involving one bit of differential amplitude coding where the coding is, as with the differential phase modulation, differential between adjacent tones. Thus, two amplitudes, A_{max} and A_{min} that are separated by $\Delta A = \alpha A_o = \alpha\{A_{max} + A_{min}\}/2$ are used.

Differential encoding the amplitude bit of tone k is accomplished by making A_k the same as A_{k-1} if $\underline{a}=0$ and changing A_k to the amplitude different from the amplitude of A_{k-1} if $\underline{a}=1$. The decoding rule for $\underline{a}$ is illustrated by Figure 5. The crosshatched region is decoded as $\underline{a}=0$, that is it is assumed that the amplitudes are the same in this region. The remaining region is decoded as $\underline{a}=1$, that is it is assumed that the amplitudes differ in this region. This rule is described by the algorithim;

"if $\{1/(1-\alpha/2)\}|Y(k-1)|\leq|Y(k)|\leq(1-\alpha/2)|Y(k-1)|$ then $\underline{a}=0$ else $\underline{a}=1$".

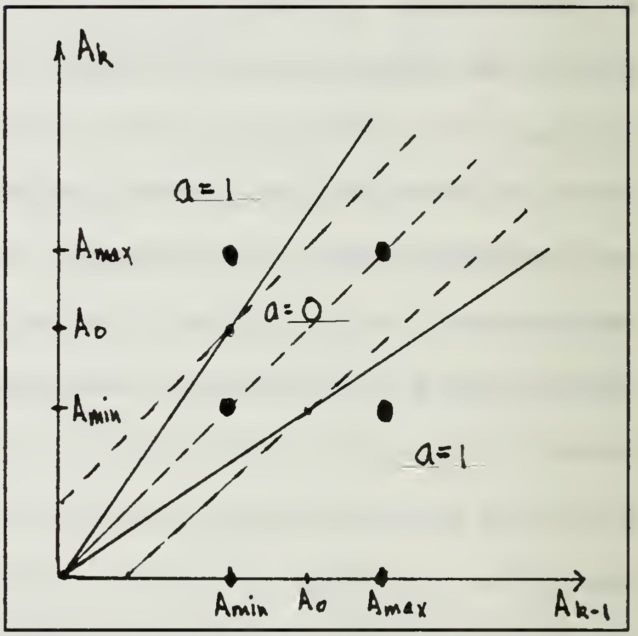

Figure 5 Differential Amplitude Decoding

Clearly the rule is independent of channel gain so long as adjacent tones have essentially identical gain.

To find symbol and bit error probabilities, the conditional probability of $|Y|$ being outside the decision regions must be computed. Recall that we have established the mean and variance of $|Y|$ and the fact that adjacent magnitudes are statistically independent. Consider the dotted lines with slope of one passing thru the points (A_o,A_{min}) and (A_{min},A_o). We shall use these as approximations for actual decision boundaries shown by the solid lines. Accordingly, it can be seen that the average symbol error probability which is also the BER is given by;

$$BER = Ps = (3/2)Q\{(1/2^{1/2})\Delta A'/2\} - (1/2)Q\{(1/2)^{1/2}3\Delta A'/2\}. \quad (4.21)$$

Comparing (4.21) and (4.17), we note that a differentially encoded amplitude bit requires approximately 3db more signal-to-noise ratio than an absolutely encoded amplitude bit in order to obtain the same BER.

V. QUADRATURE AMPLITUDE MODULATION

The purpose of this Chapter is to describe more complex modulation constellations, constellations that combine the amplitude and phase modulation schemes discussed in the previous two Chapters. As mentioned in the Introduction, we shall refer to these as "MFQAM systems" even though they will not have Z_2 (that is rectangular) grid points as is commonly found in single carrier QAM constellations.

To be more specific, we will first consider systems that utilize differential phase modulation of M bits and one bit of differential amplitude modulation. These systems will be referred to as "fully differential systems". Also, we will consider systems that use differential phase coding and absolute amplitude coding. A system of this type with M bits of differential phase modulation and N bits of absolute amplitude modulation will be referred to as an "(M,N) Hybrid".

Single Amplitude Bit MFQAM Systems

These systems accept Q=M+1 bit codewords $\underline{s}$ which are partitioned into two quad bits that determine the phase difference of adjacent tones to the nearest ninety degrees , M-2 delphase bits that are coded into phase within the quadrant and one amplitude bit the is encoded as tone amplitude. Thus a codeword is divided into the three subsets

$$\underline{s} = \underline{q}, \underline{d}, \underline{a} \ . \qquad (5.1)$$

As described in Chapters II and III, all coding is done in accordance with a Grey (reflex) code such that nearest neighbor points in QAM constellations differ in only one bit position. The BER for these constellations is determined from (2.10) using (2.11) and (2.12). We shall consider only single and double bit errors. Accordingly,

$$\Pr[B=1|\underline{s}=\underline{s}_i] \text{ and } \Pr[B=2|\underline{s}=\underline{s}_i] \qquad (5.2)$$

are to be evaluated for all $\underline{s}_i$.

a.Single Bit Errors

Single bit errors occur when an amplitude bit decoding error occurs and all the phase bits are decoded correctly or when a single bit phase error occurs and the amplitude bit is decoded correctly.

These correspond to the error patterns;

$$\underline{e} = \underline{0},1 \text{ and } \underline{e} = \underline{e}_{p1},0 \qquad (5.2)$$

where $\underline{e}_{p1}$ is a single bit phase error pattern.

Therefore the single bit error probability is,

$$\Pr[B=1] = \Pr[\underline{e}_a=1,\underline{e}_p=\underline{0}] + \Pr[\underline{e}_{p1},\underline{e}_a=0] \qquad (5.3)$$

If we assume amplitude and phase decoding errors are independent (which is certainly not strictly true but is probably a satisfactory approximation at low error rates) then,

$$\Pr[B=1] = \Pr[\underline{e}_a=1]\Pr[\underline{e}_p=\underline{0}] + \Pr[\underline{e}_{p1}]\{1-\Pr[\underline{e}_a=1]\}$$

$$\approx \Pr[\underline{e}_a=1] + \Pr[\underline{e}_{p1}] \qquad (5.4)$$

where the approximation is accurate at low symbol error rates.

The probability of an amplitude bit error is clearly independent of the phase of tones and is given by (4.17) and (4.21) for absolute and differential amplitude coding respectively.

The probability of phase bit errors is identical for all phases but depends on the amplitude of the complex rotated product which we have called M_k and whose statistics are given in (3.24). A single amplitude bit, that is two amplitude levels, results in three different complex rotated product amplitude levels for M_k depending on whether adjacent tones both have amplitudes A_{min}, both have amplitudes A_{max}, or one has amplitude A_{min} and the other amplitude A_{max}. Since we have assumed all symbol patterns are equally likely, then the first two situations each occur with probability 1/4 and the latter with probability 1/2.

Let us designate the three levels of M'_k as M'_{min}, M'_{mid} and M'_{max}. Then the probability of any particular phase bit error pattern is

$$\Pr[\underline{e}_p] = \tfrac{1}{2}\Pr[\underline{e}_p|M'_{mid}] + \tfrac{1}{4}\{\Pr[\underline{e}_p|M'_{min} + \Pr[\underline{e}_p|M'_{max}]\} \qquad (5.5)$$

The probability of single (and double) phase bit error patterns are given in (3.35) where $p(\Delta\phi)$ is given in terms of M'_k by (3.27) for MFDPSK. The three levels for M'_k are found from (3.24) for $E/N_o > 1$ to be,

$$M'_{min} = \{E_{min}/N_o\}^{1/2}$$

$$M'_{mid} = \{E_{min}E_{max}/EN_o\}^{1/2} \qquad (5.6)$$

$$M'_{max} = \{E_{max}/N_o\}^{1/2}.$$

In terms of the amplitude level spread parameter α, $E_{min}^{1/2} = E_o^{1/2}(1-\alpha/2)$, $E_{max}^{1/2} = E_o^{1/2}(1+\alpha/2)$ and E_o, the energy of an average amplitude tone is related to the average energy of the tones by (4.5). Specifically we find that,

$$M'_{min} = \{(E/N_o)(1-\alpha/2)/(1+\alpha/2)\}^{1/2}$$

$$M'_{mid} = \{(E/N_o)(1+(\alpha/2)^2)\}^{1/2} \qquad (5.7)$$

$$M'_{max} = \{(E/N_o)(1+\alpha/2)/(1-\alpha/2)\}^{1/2}.$$

Thus (5.7) in conjuction with (5.5) and (3.35) determines the probability of single phase bit errors in terms of the $\rho = SNR_{WB} = E/N_o$, the number of phases 2^M and the parameter α. Finally the BER due to single bit errors is well approximated by

$$BER_1 = Q^{-1}\{Pr[\underline{e}_a=1] + Pr[\underline{e}_{p1}]\} \tag{5.8}$$

with

$$Pr[\underline{e}_a=1] = \begin{cases} Q(\rho a_0) & \text{for hybrid coding} \\ (3/2)Q(\rho a_0/2^{1/2}) - (1/2)Q(3\rho a_0/2^{1/2}) & \text{for fully} \end{cases}$$

differential coding, and

$$Pr[\underline{e}_{p1}] = Q(\rho a_2)-Q(3\rho a_2)+\tfrac{1}{2}[Q(\rho a_1)-Q(3\rho a_1)+Q(\rho a_3)-Q(3\rho a_3)]$$

with

$$\rho = (E/N_o)^{1/2}$$

$$a_o = \alpha/\{2[1+(\alpha/2)^2]\}^{1/2}$$

$$a_1 = \sin(\pi/2^M)\{(1-\alpha/2)/(1+\alpha/2)\}^{1/2}$$

$$a_2 = \sin(\pi/2^M)\{1+(\alpha/2)^2\}^{1/2}$$

$$a_3 = \sin(\pi/2^M)\{(1+\alpha/2)/(1-\alpha/2)\}^{1/2}.$$

Examination of (5.8) shows that all the a values increase with α except a_1 which determines the phase errors due to complex products of the minimum amplitude tones. This makes sense as we know that the amplitude errors decrease with increasing amplitude difference and the large amplitude tones will have more energy with increasing amplitude differential.

One design procedure we have investigated is to choose α such that $a_1 = a_0$ for the hybid, i.e. absolute amplitude bit coding and $a_1 = a_0/2^{1/2}$ for differential amplitude coding. This leads to the following values of α;

Q	M	α Fully Differential Coding	α Hybid Coding	α Geo.
3	2	.938	.728	.88
4	3	.593	.447	.54
5	4	.335	.250	.32
6	5	.180	.130	.18

TABLE 5.1

Amplitude Spread Factor for

a Single Bit Encoded as Amplitude

From a purely geometrical point of view, values of

$\alpha=\Delta\phi/(1+\Delta\phi/2)$ space the constellation points approximately equally. These values are listed for comparison in Table 5.1 too. Using the optimum values of α yeilds the **a** coefficients shown in Table 5.2.

	a_0	a_1	a_2	a_3
<u>M=2</u>				
Differential	.601	.425	.778	1.20
Hybrid	.485	.485	.740	1.05
<u>M=3</u>				
Differential	.403	.285	.410	.525
Hybrid	.310	.310	.40	.482
<u>M=4</u>				
Differential	.232	.164	.20	.231
Hybrid	.172	.172	.20	.217
<u>M=5</u>				
Differential	.127	.090	.10	.110
Hybrid	.092	.092	.10	.107

TABLE 5.2

Error Function Coefficients

for MF-QAM

Theoretical values of $SNR_{WB} = E/N_o = \rho$ are given in db in Table 5.3 for BERs of 10^{-2}, 10^{-4} and 10^{-6} due to single bit errors for MF16-QAM. (E_b/N_o values are 6 db lower.)

	10^{-2}	10^{-4}	10^{-6}
Differential	17.9	21.9	24.2
Hybrid	15.9	21.0	23.3

TABLE 5.3

Theoretical SNRs in db for BER_1s shown for Fully Differential and Hybrid MF16-QAM

b. Double Bit Errors

Double bit errors are less likely than single bit errors but can contribute to the overall BER at the lower values of SNR_{WB}. The probability of a double bit error is

$$P[B=2] = Pr[\underline{e}_{p2}, \underline{e}_a=0] + Pr[\underline{e}_{p1}, \underline{e}_a=1]$$

$$\leq Pr[\underline{e}_{p2}] + Pr[\underline{e}_a=1] \qquad (5.9)$$

where the last term in the inequality is used as a bound because of the probable high correlation between amplitude and phase bit errors and is given by (5.8). The double phase error probability is determined from (5.5) and (3.35) with (3.27). This is always much less than the probability of an amplitude bit error when the amplitude spread has been adjusted to a value such that amplitude bit errors and single phase bit errors are approximately equal. Thus we may approximate the BER due to double bit errors as

$$BER_2 \approx 2Q^{-1}\Pr[\underline{e}_a=1] \tag{5.10}$$

with

$$\Pr[\underline{e}_a=1] = \begin{cases} Q(\rho a_0) & \text{for hybrid coding} \\ (3/2)Q(\rho a_0/2^{1/2}) - (1/2)Q(3\rho a_0/2^{1/2}) & \text{for fully} \end{cases}$$

differential coding.

The effect of double bit errors is to approximately double the BER determined by BER_1. However, because the $Q(x)$ function falls off so rapidly, this can be offset by 0.3, 0.5, and 0.8 db increases in SNR for BERs of 10^{-6}, 10^{-4}, and 10^{-6} respectively.

Multi-level Amplitude MFQAM Systems

These systems become quite complex to analyze exactly because of the multiplicity of amplitudes that occur in the complex rotated product demodulation of the differential phase terms. However, a reasonable hueristic design procedure is to set the differential amplitude level such that the distance in the constellation between the two smallest amplitudes is equal to the radial distance between adjacent phase points on the smallest amplitude ring. This is illustrated in Figure 6. The design procedure is to set

$$\Delta A' = A'_{min}\, \Delta\phi \qquad (5.11)$$

(which leads to $\alpha = \{1/\Delta\phi + (2^N-1)/2\}^{-1}$).

With this constraint, we see that the single bit error probability will be bounded by

$$P[B=1] \le Pr[\underline{e}_a=1] + Pr[\underline{e}_{p1}] \qquad (5.12)$$

which can be found from (4.3) for the amplitude bit errors and from (3.35) and (3.27) for the phase bit errors. We now use in addition the facts that

$$\Delta A' = 2\ \{2E/N_o\}^{1/2}\ f(\alpha,N) \qquad (5.13)$$

with $f(\alpha,N)$ given by (4.11) and $\Delta\phi = 2\pi/2^M$. Thus,

$$Pr[\underline{e}_a=1] = 2Q(\Delta A'/2)$$

and

$$\Pr[\underline{e}_{p1}] = 2Q(M'_{min}\,\Delta\phi/2)$$

with $M'_{min} = (E_{min}/N_o)^{1/2} = A'_{min}/2^{1/2} = \Delta A'/2^{1/2}\Delta\phi$ such that

$$\Pr[\underline{e}_{p1}] = 2Q\{\Delta A'/(2(2^{1/2}))\} \qquad (5.14)$$

It can be noted that the phase errors are 3 db worse than the amplitude errors because of the differential coding. However, the amplitude errors are the same for all amplitude levels whereas the phase errors are much less for the larger amplitude values. Using this procedure one obtains the parameters of Table 5.4 for the (M,N) hybrids indicated.

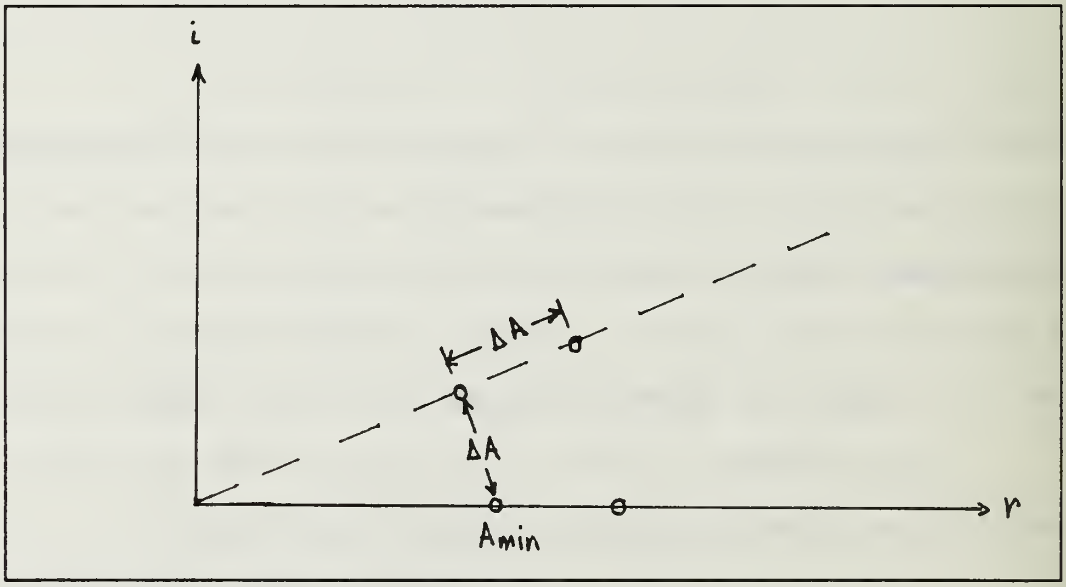

Figure 6
Ring Spacing to Equalize Amplitude and Phase Bit Errors

(M,N)	α	$f(\alpha,N)$ = $\Delta A'/(2(2^{1/2})\rho)$	SNR in db (BER) $10^{-6}, 10^{-4}, 10^{-2}$
(3,1)	.563	.28	24.8
(4,2)	.248	.122	32.0
(5,2)	.152	.075	36.3 ,34.3,31.0
(5,3)	.116	.058	38.5
(6,2)	.085	.046	40.5

TABLE 5.4

SNRs Required for Single BER

shown for Hybrids from 4 to 8 Bits/Hz.

The (3,1) 16-QAM case has been included for comparison. Notice how these geometrical arguments lead to results very close to those obtained in the previous section by more thourough analysis. (24.8 vice 23.3 db for 10^{-6} BER and α of .563 vice .447). This gives us considerable confidence in our other results. The (5,2) hybrid BERs also seem to be closely verified by our current experiments.

VI DISCUSSION AND CONCLUSIONS

The principal objective of this report was to document the theortical predictions for various types of differential coding in the frequency domain for Multi-Frequency Modulation(MFM). In order to do this in a systematic fashion, general results were derived for probability of symbol error and bit error rate(BER) in Chapter II in terms of $\underline{e}$, the error pattern vector. The coding we have developed partitions a symbol $\underline{s}$ into phase bits $\underline{p}$ and amplitude bits $\underline{a}$. Phase bits are further partitioned into two quad bits $\underline{q}$ that determine the phase quadrant and any remaining phase bits that determine tone phase within a quadrant.

In Chapter III results were presented for coherent phase modulation with maximum likelihood decoding and for differential phase modulation using the "complex product rotation" algorithim followed by maximum likelihood decoding. In order to minimize BER, the phase states are Grey coded so that the most probable symbol errors result only in single bit errors.

Differential phase coding and decoding using complex rotated product results in a theoretical 3 db loss of performance versus SNR (or E_b/N_o) when considering detection on a tone by tone basis. A well known phenomena in differential coding is the increased probability of error pairs[8]. This occurs because the noise present in a single tone affects the complex rotated product for two adjacent symbols. Thus if noise is high in one tone, there is an increased probability of two adjacent symbol decoding errors. We have not presented a theoretical treatment of this phenomena in this report but will in a subsequent report that treats interleaving and forward error control techniques, including trellis coding, for MFDM. However, it is mentioned here because in addition to the 3 db of performance loss, error pairing is another penalty paid for differential coding.

In Chapter IV results were presented for amplitude coding the tones of MFM. Multi-level amplitudes with equally spaced intervals were assumed. The performance of maximum likelihood detection of the amplitude interval of each tone at the receiver was given. This technique is of course incoherent because it does not require reference phases for the tones for decoding, but it does require knowledge of the absolute gain of the channel at each frequency in order to set the decoding thresholds. Consequently, we also presented results for differential coding/detection of a single amplitude bit. Again, about a 3 db penalty is paid for differential coding and the increased probability of error pairing can be expected here as well.

In Chapter V the concepts of phase and amplitude coding were combined to create MFQAM constellations. All of these constellations are circularily symmetric and because their phase is differentially encoded, the decoding is independent of absolute tone phase shift thru the channel. Constellations with a single differentially encoded amplitude bit and differential phase are termed "fully differential" constellations. Constellations with absolute level decoding of amplitude bits but differential decoding of the phase bits are termed "hybrid" constellations. SNR required to achieve BER of 10^{-6} were given for a number of the constellations in Table 5.4. Constellation design procedures were presented that purport to very nearly minimize BER.

The SNRs (or E_b/N_0) required for differential constellations to achieve given levels of BER are about 3 to 5 db higher than is required for fully coherent MFM using rectangular (Z_2) grid for locating the tone real and imaginary parts. The advantage gained by differential coding is the removal of any requirement to equalize the channel and to determine reference phases for each of the tones. In certain applications, the penalities of differential coding in SNR will be outweighed by its simplicity of implementation, synchronization and robustness in the face of changing channel parameters. In other applications, it will not be worth the penalty.

The purpose of this report was to present design procedures and performance expectations for frequency domain differentially encoded MFM with signalling efficiencies of 2 to 8 bits/Hz.
One of the newest techniques of error control in single carrier digital modulation is trellis coded modulation(TCM)[10]. Our direct funded research project is now directed toward applying TCM to MFM by trellis coding in the frequency domain. In brief, we believe that TCM has the potential for improving our raw E_b/N_0 performance by 3 to 4 db. Furthermore, with interleaving it is hoped that the effects of differentially induced error pairs can be removed.

Application of TCM to pure phase coding is believed to be relatively straighforward. From an implementaion point of view, the

MFM receiver must be equipped with a Viterbi decoder. Although complex, the procedures are well established to do this and Viterbi decoders for TCM have been written for the Motorola 56000 DSP chip [11]. Trellis coding the QAM constellations analyzed in Chapter V may not be a straightforward application of known results. Previous work has concentrated on TCM for Z_2 grids[10] and the conversion to our circularly symmetric constellations is not directly apparent. This problem is at the top of our current research agenda.

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